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Adsorption of uniform lattice animals with specified topology

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Abstract. We study the effects of topology on the free energy of uniform lattice animals interacting with a surface. Topology is specified by fixing an abstract graph, τ, and the lattice animals considered are embeddings of τ in the square and simple cubic lattice. We prove that such embeddings in the simple cubic lattice and interacting with a plane have the same free energy as self-avoiding walks independent of the choice of τ and independent of whether or not the embeddings are restricted to be uniform. For embeddings in the square lattice and interacting with a line, we prove that the free energy may differ from that for walks depending on whether τ has a cut edge. Further restricting the embeddings to be uniform forces the free energy to be different from that for walks for all τ (except the τ corresponding to walks) and we obtain bounds on the free energy which depend on the number of branches, cycles and vertices of degree 3 and 4 in the graph.

1. Introduction

In 1979 Lubensky and Isaacson proposed a lattice animal model of randomly branched polymers with excluded volume. A lattice animal is any finite subgraph of a lattice. Since 1979 interest has developed in modelling branched polymers with specified topologies using lattice animals (Miyake and Freed 1983, Gaunt *et al* 1984, Lipson *et al* 1987, Soteros and Whittington 1989). The goal of such models is to predict the effects of branching on the properties of polymers with excluded volume. Generally the approach is to find relationships between a model of a polymer with specified topology and the self-avoiding walk (SAW) model for linear polymers. In particular, Gaunt *et al* (1984) have proved that the growth constant for lattice animals on the d-dimensional hypercubic lattice with cyclomatic index c and n_k vertices of degree k ($k=3,\ldots,2d$) is the same as that for SAWs. Most recently Soteros *et al* (1991) have proved the more general result that the growth constant for lattice animals which have a fixed topology specified by a graph (i.e. lattice animals homeomorphic to a fixed abstract graph) is also the same as that for SAWs.

Similarly, lattice animals can be used to study uniform branched polymers (every branch is composed of the same number of monomers). Soteros and Whittington (1989) have proved that uniform brushes have the same growth constant as SAWs. However, the more general question of whether uniform lattice animals homeomorphic to a fixed abstract graph have the same growth constant as SAWs remained open. In this paper it is proved that for two and three dimensions such lattice animals do have the same growth constant as SAWs. These results cannot be proved by extending the uniform brush proof; instead a new general argument is required. The result in three dimensions is a corollary of a result we obtain for uniform branched polymers interacting with a surface. The proof of the result in two dimensions is separate.

Hammersley *et al* (1982) have studied the properties of the free energy of SAWs interacting with a surface. Recent results indicate that at least when the dimension of the lattice is greater than 2, uniform lattice animals with specified topologies and interacting with a surface have the same free energy as SAWs interacting with a surface. This has been proved for uniform stars (Whittington and Soteros 1991), uniform brushes (Zhao and Lookman 1991a) and k-loops (Zhao and Lookman 1991b). In this paper we prove that it is true for uniform lattice animals with any specified topology. The proof of this for any topology is not an extension of the proof for the uniform star, uniform brush or uniform k-loop case. Instead it is first necessary to obtain new results about self-avoiding polygons (a model of ring polymers) in wedges. Then these results are combined with some results from graph theory to obtain the final general result. The concatenation arguments used in the proof are general enough that they can be applied to almost any system represented by a finite number of non-intersecting subgraphs of a regular lattice with dimension at least 3. In addition most previous results concerning the growth constant and adsorption free energy of branched polymers with specified topology can now be obtained as corollaries of this result.

In two dimensions, Whittington and Soteros's results for uniform stars indicate there is a marked difference between the behaviour of uniform branched polymers interacting with a surface when compared to the behaviour of linear polymers interacting with a surface. This is due to the fact that some of the branches of the polymer are prevented from having any contact with the surface by other branches of the polymer. We explore the 2D case further in this paper. Here it is shown that the result for uniform stars generalizes to any uniform branched polymer with specified topology, i.e. there is a shading effect which causes the interaction of the polymer to be different from that of a linear polymer. We prove this by obtaining bounds dependent on c, n_3 and n_4 for the free energy of a branched polymer with specified topology. We also show that in the case that the branched polymer is not restricted to being uniform, the free energy of the polymer only depends on whether the specified topology of the polymer contains a cut edge. If the topology has a cut edge the polymer's free energy is the same as that of a linear polymer's; if it does not have a cut edge the free energy is the same as that of a ring polymer. In particular this implies that the free energy of a dumbbell-shaped polymer is different from the free energy of a polymer shaped like the Greek letter θ even though c, n_3 and n_4 are the same for both.

The results discussed above are presented in detail in the next three sections of this paper. In the first section we review the required definitions and required results from the theory of SAWs. In the second section the proofs of the results concerning the simple cubic lattice are presented. In the third section the square lattice results are proved.

2. Properties of SAWs

We are primarily concerned with lattice animals which are connected subgraphs of the square (Z^2) and simple cubic (Z^3) lattices. A subgraph of Z^d for $d = 2$ or $d = 3$ is composed of lattice points, called *vertices*, and lattice edges. The *degree* of a vertex in the subgraph is defined to be the number of edges of the lattice incident on the vertex. We also assume that on Z^d a vertex has integer coordinates $(x_1, \ldots, x_d)$ and we let $\hat{u}_j$ be the jth unit vector.

An *n-step self-avoiding walk* (or n-SAW) beginning at lattice point z_0 is an $(n+1)$-tuple of distinct lattice points $(z_0, \ldots, z_n)$ where z_i and z_{i+1} are adjacent in the lattice, $0 \leqslant i < n$ and the coordinates of z_i are $(x_1^{(i)}, \ldots, x_d^{(d)})$. The n-SAW α is *rooted* if $z_0 = 0$. For each n, let $c_n^{(d)}$ denote the number of distinct (as $(n+1)$-tuples) rooted n-SAWs in Z^d, $d = 2$ or 3. Then Hammersley and Morton (1954) have shown that

$$0 < \lim_{n\to\infty} n^{-1} \log c_n^{(d)} = \kappa_d < \infty \tag{2.1}$$

where κ_d is called the *connective constant* for Z^d. An *n-step self-avoiding polygon* (or n-SAP) is any connected subgraph of the lattice composed of n edges and n vertices in which each vertex has degree 2. Two SAPs are equivalent if one is a translate of the other. We write $p_n^{(d)}$ for the number of inequivalent n-SAPs in Z^d; $p_n^{(d)}$ is zero if n is odd so we adopt the convention that n is even in any statement involving $p_n^{(d)}$. Hammersley (1961) has shown that

$$\lim_{n\to\infty} n^{-1} \log p_n^{(d)} = \kappa_d. \tag{2.2}$$

Let $\mathscr{G}$ represent the union of the circle graph (the connected graph with exactly one vertex and one edge) and the set of abstract finite connected graphs with no vertices of degree 2. We define a *branch point* of a graph $\tau \in \mathscr{G}$ as a vertex of degree greater than 2 and an *end point* as a vertex of degree 1. A *branch* is defined as an edge or set of edges either between two branch points, two end points, or a branch point and an end point, which does not contain any other branch or end point.

Let $\mathscr{G}_k \subset \mathscr{G}$ be the set of graphs in $\mathscr{G}$ having maximum vertex degree less than or equal to k. Consider $\tau \in \mathscr{G}_{2d}$. An *embedding* of τ in Z^d will be any finite subgraph of Z^d which is homeomorphic to τ. Hence an embedding of τ in Z^d is any lattice animal in Z^d which is homeomorphic to τ. We refer to the number (n) of occupied lattice vertices of an embedding in Z^d as the *size* of the embedding, and consider identical those embeddings which are superimposable by translation.

Let $g^{(d)}(n, \tau)$ be the number of embeddings of the graph $\tau \in \mathscr{G}_{2d}$ in Z^d of size n. For instance, if τ corresponds to the circle graph we write $\tau = \pi$ and then $g^{(d)}(n, \pi) = 0$ if n is odd, $g^{(d)}(n, \pi) = p_n^{(d)}$ for n even. Therefore equation (2.2) can be rewritten

$$\lim_{n\to\infty} n^{-1} \log g^{(d)}(n, \pi) = \kappa_d \tag{2.3}$$

where n goes to infinity through the even numbers. If $\tau \in \mathscr{G}_{2d}$ corresponds to the graph with exactly two vertices, each of degree 1, and one edge, then we write $\tau = \omega$ and equation (2.1) implies

$$\lim_{n\to\infty} n^{-1} \log g^{(d)}(n, \omega) = \kappa_d. \tag{2.4}$$

Soteros *et al* (1991) have shown that

$$\lim_{n\to\infty} n^{-1} \log g^{(3)}(n, \tau) = \kappa_3 \tag{2.5}$$

for any $\tau \in \mathscr{G}_6$ and where n is assumed to go to infinity through even integers if $g^{(3)}(n, \tau) = 0$ for n odd. In general we define the limit, if it exists, $\lim_{n\to\infty} n^{-1} \log g^{(d)}(n, \tau)$ to be the *growth constant* for τ in Z^d. Equation (2.5) thus indicates that the growth constant for any $\tau \in \mathscr{G}_6$ in Z^3 is the same as the growth constant for SAWs in Z^3.

Since we are interested in modelling polymers interacting with a surface it is useful to review here the results that are known about models of linear polymers interacting with a surface. Hammersley *et al* (1982) modelled a linear polymer interacting with a surface using a SAW in the half-space $x_d \geq 0$ interacting with the surface $x_d = 0$. In particular, let H^d represent the half-space $x_d \geq 0$ in Z^d. Let $c_{n,m}^{(d)}$ denote the number of distinct rooted n-SAWs in H^d with $m+1$ vertices in the hyperplane $L^d = \{(x_1, \ldots, x_d) \in H^d | x_d = 0\}$. Define the generating function

$$A_n^{(d)}(\beta) = \sum_{m=0}^{n} c_{n,m}^{(d)} \, e^{m\beta}. \tag{2.6}$$

Hammersley *et al* (1982) have shown that the limit, hereafter called the *free energy*,

$$A^{(d)}(\beta) = \lim_{n\to\infty} n^{-1} \log A_n^{(d)}(\beta) \tag{2.7}$$

exists for all β and that

$$\max(\kappa_d, \kappa_{d-1} + \beta) \leq A^{(d)}(\beta) \leq \max(\kappa_d, \kappa_d + \beta). \tag{2.8}$$

From this they conclude that there is a phase transition in the model (corresponding to adsorption) for some critical value of β, β_c, where $0 < \beta_c \leq \kappa_d - \kappa_{d-1}$.

Finally, we will need to use some known results about models of linear polymers confined to wedges. Hammersley and Whittington (1985) modelled linear polymers confined to a wedge using SAWs on Z^d confined to a wedge. Define an (α, β, T)-wedge for $\alpha < \beta$ to be $\{(x_1, \ldots, x_d) \in Z^d | 0 \leq x_1, \alpha x_1 \leq x_2 \leq \beta x_1 + T\}$. Note that a $(0, \alpha, 0)$-wedge is equivalent to a $(1/\alpha, \infty, 0)$-wedge. Hammersley and Whittington (1985) proved that rooted SAWs in a $(0, \alpha, 0)$-wedge in Z^d have growth constant κ_d. Define $c_{n,m}^{\alpha,\beta}$ to be the number of rooted n-SAWs in an $(\alpha, \beta, 0)$-wedge in H^3 with $m+1$ vertices in L^3, $m \geq 0$, and such that $0 = x_1^{(0)} \leq x_1^{(i)} < x_1^{(n)}$ $(i = 1, \ldots, n-1)$. Let $A_n^{\alpha,\beta}(\varepsilon) = \sum_{m=0}^{n} c_{n,m}^{\alpha,\beta} \, e^{\varepsilon m}$. An argument given by Whittington (1988) which proves that rooted SAWs in $(0, \alpha, 0)$-wedges have connective constant κ_d can be extended to prove that

$$\lim_{n\to\infty} n^{-1} \log A_n^{0,\alpha}(\varepsilon) = A^{(d)}(\varepsilon). \tag{2.9}$$

3. The results for the simple cubic lattice

In this section we prove for any graph τ that uniform embeddings of τ in H^3 interacting with L^3 have the same free energy as SAWs in H^3 interacting with L^3. To prove the result we start by showing that it is true for the case $\tau = \pi$, i.e. the case that the embeddings are SAPs. We then show that SAPs confined to certain wedges have the same free energy as unconstrained SAWs. This wedge result combined with some graph theory allows us to prove the result for any τ.

First we need a few more definitions. For any set S_0 of vertices we define the top (bottom) vertex as follows. First construct the subset $S_1 \subset S_0$ such that the coordinate x_1 of every vertex in S_1 has the maximum (minimum) value over all vertices in S_0. We then recursively construct $S_k \subset S_{k-1}$ such that the coordinate x_k of every vertex in S_k has the maximum (minimum) value over all vertices in S_{k-1}. Let j be the smallest integer such that S_j contains precisely one vertex, and call this vertex v_t (v_b), the top (bottom) vertex of S_0.

Let $p^{+}_{n,m}(d)$ denote the number of distinct SAPs in H^d with a total of $m+1$ vertices in L^d, $m \geq 0$ (two polygons are equivalent if they are superimposable by translation). Let $p^{++}_{n,m}(d)$ denote the number of distinct SAPs in H^d with v_b in L^d and a total of $m+1$ vertices in L^d, $m \geq 0$. Clearly

$$p^{++}_{n,m}(d) \leq p^{+}_{n,m}(d) \leq c^{(d)}_{n-1,m}. \tag{3.1}$$

In addition, for $d \geq 2$ we can define

$$B^{+(d)}_n(\beta) = \sum_{m=0}^{n} p^{+}_{n,m}(d)\, e^{\beta m} \tag{3.2}$$

and

$$B^{++(d)}_n(\beta) = \sum_{m=0}^{n} p^{++}_{n,m}(d)\, e^{\beta m}. \tag{3.3}$$

Lemma 1.

$$\lim_{n\to\infty} n^{-1} \log B^{++(3)}_n(\beta) = \lim_{n\to\infty} n^{-1} \log B^{+(3)}_n(\beta) = A^{(3)}(\beta). \tag{3.4}$$

Proof. We note that the proof in the literature of equation (2.2) is not easily extended to a proof of equation (3.4). Instead, to prove equation (3.4) we note the following. Hammersley *et al* (1982) showed that a rooted n-SAW $(z_0, \ldots, z_n)$ in H^d with $z_i = (x^{(i)}_1, \ldots, x^{(i)}_d)$ and satisfying $0 = x^{(0)}_1 \leq x^{(i)}_1 < x^{(n)}_1$ $(i = 1, \ldots, n-1)$, $0 = x^{(0)}_d = x^{(n)}_d$ has free energy $A^{(d)}(\beta)$. (In other words, SAWs in H^d 'unfolded' in the x_1 direction and which return to L^d at their last step have the same free energy as all SAWs.) They proved this by unfolding ordinary SAWs so that a SAW satisfying $0 = x^{(0)}_1 \leq x^{(i)}_1 < x^{(n)}_1$ results. They then concatenated in pairs unfolded walks ending at the same vertex so as to form an unfolded walk which has its last vertex in L^d. We will refer to the resulting SAWs as (*)-walks. M Hammersley (1987) pointed out that these (*)-walks can be concatenated in pairs (one above L^d and the other below L^d) to form a SAP and this construction provides an alternative proof of equation (2.2). Analogously for $d > 2$, one can concatenate two (**)-walks (one in the quarter-space $0 \leq x_1$, $0 \leq x_d$, $0 \leq x_2$ and the other in the quarter-space $0 \leq x_1$, $0 \leq x_d$, $x_2 \leq 0$) to create a SAP with its bottom vertex in L^d. Since (**)-walks have free energy $A^{(3)}(\beta)$ this construction implies

$$A^{(3)}(\beta) \leq \liminf_{n\to\infty} n^{-1} \log B^{++(3)}_n(\beta) \leq \liminf_{n\to\infty} n^{-1} \log B^{+(3)}_n(\beta). \tag{3.5}$$

Multiplying equation (3.1) with $d = 3$ by $e^{\beta m}$, summing over m, taking logarithms, dividing by n and letting n go to infinity gives

$$\limsup_{n\to\infty} n^{-1} \log B^{++(3)}_n(\beta) \leq \limsup_{n\to\infty} n^{-1} \log B^{+(3)}_n(\beta) \leq A^{(3)}(\beta). \tag{3.6}$$

Equations (3.5) and (3.6) give equation (3.4). □

We next look at the special case of SAPs in wedges and start by making some definitions and extending some results about the growth constant for SAPs and SAWs in wedges to similar results for their free energies. We then prove that the free energy of a SAP in an (α, β, T)-wedge is $A^{(3)}(\beta)$.

For $1 \leq M \leq n/2 - 1$, $d \geq 3$, define $p^{\alpha,M,d}_{n,m}$ to be the number of n-SAPs in the subset of a $(0, \alpha, 0)$-wedge $W^{\alpha}_M = \{(x_1, x_2, \ldots, x_d) \in H^d \mid 0 \leq x_1 \leq M,\ 0 \leq x_2 \leq \min\{\alpha x_1, M - x_1\}\}$ such that v_b is the origin, $v_t = (M, 0, \ldots, 0, 1)$, $v_b + \hat{u}_d$ and $v_t - \hat{u}_d$

are vertices of the polygon, $m+1$ vertices of the polygon lie in L^d, $m\geq 1$, and the only vertices of the polygon in the plane $x_2=M-x_1$ are v_t and $v_t-\hat{u}_d$.

Let $\mathcal{P}_{n,m}^{\alpha,M,d}$ represent the set of n-SAPs in W_M^α. Let

$$B_n^{\alpha,M,d}(\varepsilon)=\sum_{m=1}^{n} p_{n,m}^{\alpha,M,d}\,\mathrm{e}^{\varepsilon m}. \tag{3.7}$$

The argument of Hammersley and Whittington (1985) for SAPs in $(0,\alpha,0)$-wedges can be extended to prove that

$$\lim_{n\to\infty} n^{-1}\log\sum_{M=1}^{n/2-1} B_n^{\alpha,M,d}(\varepsilon)=A^{(3)}(\varepsilon)\qquad d=3. \tag{3.8}$$

Similarly, for $M\leq n$, define $c_{n,m}^{\alpha,M,d}$ to be the number of rooted n-step SAWs in W_M^α such that the SAW ends at $(M,0,\ldots,0)$, has $m+1$ vertices in L^d, $m\geq 1$, and $x_2^{(i)}<M-x_1^{(i)}$ for $i<n$. Let

$$A_n^{\alpha,M,d}(\varepsilon)=\sum_{m=1}^{n} c_{n,m}^{\alpha,M,d}\,\mathrm{e}^{\varepsilon m} \tag{3.9}$$

and

$$A_n^{\alpha,d}(\varepsilon)=\sum_{M=1}^{n} A_n^{\alpha,M,d}(\varepsilon). \tag{3.10}$$

The argument of Whittington (1988) for SAWs in $(0,\alpha,0)$-wedges can be extended to prove that

$$\lim_{n\to\infty} n^{-1}\log A_n^{\alpha,d}(\varepsilon)=A^{(d)}(\varepsilon)\qquad d=2,3. \tag{3.11}$$

For $d\geq 3$, concatenating an n_1-step SAP in $W_{M_1}^\alpha$ to an n_2-step SAP in $W_{M_2}^\alpha$ by superimposing two vertices of each and deleting two edges leads to

$$p_{n_1,m_1}^{\alpha,M_1,d}\,p_{n_2,m_2}^{\alpha,M_2,d}\leq p_{n_1+n_2-2,m_1+m_2}^{\alpha,M_1+M_2,d}\leq p_{n_1+n_2,m_1+m_2}^{\alpha,M_1+M_2,d}. \tag{3.12}$$

Hence

$$B_{n_1}^{\alpha,M_1,d}(\varepsilon)B_{n_2}^{\alpha,M_2,d}(\varepsilon)\leq(n_1+n_2+1)B_{n_1+n_2}^{\alpha,M_1+M_2,d}(\varepsilon). \tag{3.13}$$

Further, for $0<\alpha<\beta$

$$p_{N,m}^{\alpha,M}\leq p_{N,m}^{\beta,M} \tag{3.14}$$

and hence

$$B_N^{\alpha,M,d}(\varepsilon)\leq B_N^{\beta,M,d}(\varepsilon). \tag{3.15}$$

Lemma 2. Given any positive numbers α and β, let $T=\lceil\alpha\rceil+1$ and $\alpha<\beta$. Define $p_{n,m}^{\alpha,\beta}$ to be the number of n-SAPs in an (α,β,T)-wedge in H^3 with $v_b=(0,0,0)$ and $m+1$ vertices in L^3. Let $B_n^{\alpha,\beta}(\varepsilon)=\sum_{m=0}^{n} p_{n,m}^{\alpha,\beta}\,\mathrm{e}^{\varepsilon m}$ then if either α or $1/\alpha$ is an integer

$$\lim_{n\to\infty} n^{-1}\log B_n^{\alpha,\beta}(\varepsilon)=A^{(3)}(\varepsilon). \tag{3.16}$$

Proof. Hammersley and Whittington (1985) have proved that SAPs in a $(0, \alpha, 0)$-wedge have growth constant κ_d. They concatenated polygons in snug boxes to get a lower bound. For this lemma, instead of a snug box we use a new construction involving snug $(0, \delta, 0)$-wedges, W_M^δ.

Let $p_{n,m}^{**}$ be the number of n-SAPs in a $(\alpha, \infty, 0)$-wedge in H^3 with v_b at the origin and v_t in the line $\{x_2 = \alpha x_1, x_3 = 1\}$ and $m+1$ vertices, $m \geq 0$, in L^3. We start by showing that such (**)-polygons have free energy $A^{(3)}(\beta)$.

Let N be such that αN is an integer and both αN and N are even. Let M be such that αM is an integer. One can concatenate a polygon in the set $\mathscr{P}_{N,m_1}^{\alpha,M,3}$ to a polygon in the set $\mathscr{P}_{\alpha N,m_2}^{1/\alpha,\alpha M,3}$ to create a polygon of size $(\alpha+1)N-2$ which starts and ends on the plane $x_2 = \alpha x_1$. The concatenation is done by rotating and reflecting the $(0, 1/\alpha, 0)$-wedge, $W_{\alpha M}^{1/\alpha}$, so that it is an $(\alpha, \infty, 0)$-wedge, then reflect the $(0, \alpha, 0)$-wedge, W_M^α, through the x_1 and x_2 axes, and finally translate the wedges so that they intersect at the plane $(x_1, \alpha M - x_1, x_3)$. Deleting an edge in each polygon creates one polygon of size $(\alpha+1)N-2$. Two edges can be added to the top of this polygon to give a polygon of size $(\alpha+1)N$. The resulting polygon is in the union of two wedges (see figure 1), W_M^α and $W_{\alpha M}^{1/\alpha}$, for $1 \leq M \leq N$ and $1 \leq \alpha M \leq \alpha N$. Any SAP created this way is a (**)-polygon. Therefore

$$\begin{aligned} p_{N,m_1}^{\alpha,M,3}\, p_{\alpha N,m_2}^{1/\alpha,\alpha M,3} &\leq p_{(\alpha+1)N,m_1+m_2}^{**} \\ &\leq p_{(\alpha+1)N,m_1+m_2}^{++} \end{aligned} \tag{3.17}$$

and hence, for any M such that αM is an integer

$$\begin{aligned} B_N^{\alpha,M,3}(\varepsilon) B_{\alpha N}^{1/\alpha,\alpha M,3}(\varepsilon) &\leq ((\alpha+1)N+1) B_{(\alpha+1)N}^{**}(\varepsilon) \\ &\leq ((\alpha+1)N+1) B_{(\alpha+1)N}^{++}(\varepsilon) \end{aligned} \tag{3.18}$$

where

$$\sum_{m=0}^{n} p_{n,m}^{**}\, e^{\varepsilon m} = B_n^{**}(\varepsilon). \tag{3.19}$$

A subadditivity argument shows that $\lim_{n\to\infty} n^{-1} \log B_n^{**}(\varepsilon)$ exists.

Let $\delta = \max\{\alpha, 1/\alpha\}$, so that $0 < 1/\delta \leq \delta$. Equation (3.15) implies

$$B_N^{1/\delta,M,d}(\varepsilon) \leq B_N^{\delta,M,d}(\varepsilon). \tag{3.20}$$

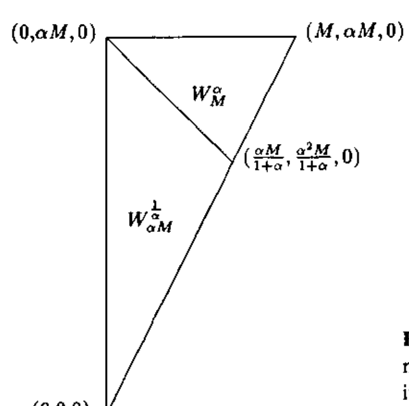

Figure 1. The union of a $W_{\alpha M}^{1/\alpha}$ and a W_M^α wedge as needed for the proof of lemma 2 is shown here as its projection in the $x^3 = 0$ plane.

If $\delta=\alpha$ let $n=N$ and $m=M$ in (3.18). If $\delta=1/\alpha$ let $n=\alpha N$ and $m=\alpha M$ in (3.18). (Note that, as long as δ is an integer, m can take on all integer values between 0 and $n/2-1$.) In either case equation (3.18) becomes

$$B_n^{\delta,m,3}(\varepsilon)B_{\delta n}^{1/\delta,\delta m,3}(\varepsilon)\leqslant((\delta+1)n+1)B^{**}_{(\delta+1)n}(\varepsilon)$$
$$\leqslant((\delta+1)n+1)B^{++}_{(\delta+1)n}(\varepsilon). \tag{3.21}$$

Equations (3.20) and (3.13) thus give for δ an integer

$$[B_n^{1/\delta,m,3}(\varepsilon)]^{\delta+1}\leqslant((\delta+1)n+1)B^{**}_{(\delta+1)n}(\varepsilon)$$
$$\leqslant((\delta+1)n+1)B^{++}_{(\delta+1)n}(\varepsilon). \tag{3.22}$$

For any n there are $n/2-1$ possible values of m and hence there exists $0<m^*\leqslant n/2-1$ such that

$$B_n^{1/\delta,m^*,3}(\varepsilon)\geqslant\frac{\sum_{m=1}^{n/2-1}B_n^{1/\delta,m,3}}{n/2-1}. \tag{3.23}$$

Let $m=m^*$ in equation (3.22) and thus equation (3.23) implies

$$\left[\frac{\sum_{m=1}^{n/2-1}B_n^{1/\delta,m,3}}{n/2-1}\right]^{\delta+1}\leqslant((\delta+1)n+1)B^{**}_{(\delta+1)n}(\varepsilon)$$
$$\leqslant((\delta+1)n+1)B^{++}_{(\delta+1)n}(\varepsilon). \tag{3.24}$$

Taking logarithms, dividing by $(\delta+1)n$, and letting n go to infinity in equation (3.24) gives

$$\lim_{n\to\infty}n^{-1}\log B_n^{**}(\varepsilon)=A^{(3)}(\varepsilon). \tag{3.25}$$

In other words, if either $1/\alpha$ or α is an integer then SAPs which have v_b at the origin and v_t in the line $\{x_2=\alpha x_1, x_3=1\}$ in a $(\alpha,\infty,0)$-wedge in H^3 have the same free energy as SAWs in H^3. (Equivalently, SAPs which have v_b at the origin and v_t in the line $\{x_2=\alpha x_1, x_3=1\}$ in a $(0,\alpha,0)$-wedge in H^3 have the same free energy as SAWs in H^3.)

Given n, let $x_1^*=k\lceil 1/\alpha\rceil$ for some k and assume $x_1^*\geqslant\lceil n/2(\beta-\alpha)\rceil$. An (α,β,T)-wedge can then contain any n-step (**)-polygon translated so that v_b is at a lattice point of the form $(x_1,\alpha x_1,0)$ for $x_1\geqslant x_1^*$. Hence as many (**)-polygons of size n as one likes can be concatenated and contained in the wedge. The (**)-polygon with $v_b=(x_1^*,\alpha x_1^*,0)$ can be connected to the bottom vertex of the (α,β,T)-wedge by concatenating the following polygon:

$$\pi_0=\{\hat u_3, k(\lceil\alpha\rceil\hat u_2,\lceil 1/\alpha\rceil\hat u_1),-\hat u_3,k(-\lceil\alpha\rceil\hat u_2,-\lceil 1/\alpha\rceil\hat u_1)\}.$$

Let A be the number of vertices in π_0 and hence π_0 has $A/2$ vertices in L^3. Concatenate r (**)-polygons each with $n=(\alpha+1)N$ vertices $m+1$ of which are in L^3 and then concatenate π_0. This results in a polygon in an (α,β,T)-wedge with $A+rn$ vertices, $A/2+rm$ of which are in L^3. Hence

$$[p^{**}_{n,m}]^r\leqslant p^{\alpha,\beta}_{nr+A,rm-1+A/2}\leqslant p^{++}_{nr+A,rm-1+A/2} \tag{3.26}$$

and by Hölder's inequality

$$[B_n^{**}(\varepsilon)]^r\leqslant(n+1)^{r-1}\sum_{m=0}^{n}(p^{**}_{n,m})^r\,\mathrm{e}^{\varepsilon rm}. \tag{3.27}$$

Thus equations (3.25), (3.4), (3.26) and (3.27) imply (3.16). □

We now show how to use the above lemma to prove that uniform embeddings of a graph in H^3 have the same free energy as SAWs in H^3. We first must define how the polymer is attached to the surface and we consider two possible cases. In the first case each branch of the polymer has at least one contact with the surface (however, we do not specify where the contact is). In the second case the whole polymer is only required to have one contact with the surface and again we do not specify where the contact is made. The results presented here can be easily modified to apply to the case where the location(s) of the initial polymer contact(s) with the surface is(are) specified.

For any $\tau \in \mathscr{G}_6$ let f be the number of branches in τ, c its cyclomatic index and n_i the number of vertices of degree i, $i \neq 2$. Define $g^{++}(n, m, \tau)$ to be the number of uniform embeddings of τ in H^3 with n edges in each branch and $m+f$ vertices in L^3 such that one vertex of each branch is in L^3. Define $g^{+}(n, m, \tau)$ to be the number of uniform embeddings of τ in H^3 with n edges in each branch and $m+1$ vertices in L^3, $m \geqslant 0$. Define $A_N^{++}(\tau, \beta) = \sum_{m=0}^{N} g^{++}(n, m, \tau)\, e^{\beta m}$ where $N = nf - c + 1$ is the total number of vertices in a uniform embedding of τ. Define $A_N^{+}(\tau, \beta) = \sum_{m=0}^{N} g^{+}(n, m, \tau)\, e^{\beta m}$.

Note that the result of Hammersley *et al* (1982) is for SAWs rooted at the origin. We will need to show that their result also applies to undirected SAWs (i.e. embeddings of $\tau = \omega$) not necessarily rooted at the origin but with at least one vertex in L^d. In fact it is straightforward to obtain upper and lower bounds for $g^{+}(n, m, \omega) = g^{++}(n, m, \omega)$ in terms of $c_{n,m}^{(3)}$ and such bounds lead to a proof of the following lemma.

Lemma 3.

$$\lim_{n\to\infty} N^{-1} \log A_N^{++}(\omega, \beta) = \lim_{n\to\infty} N^{-1} \log A_N^{+}(\omega, \beta) = A^{(3)}(\beta) \tag{3.28}$$

where $N = n+1$.

Now we can obtain the main result of this section.

Theorem 1. For any $\tau \in \mathscr{G}_6$,

$$\lim_{n\to\infty} N^{-1} \log A_N^{++}(\tau, \beta) = A^{(3)}(\beta) \tag{3.29}$$

with $N = nf - c + 1$ and where if τ is not bipartite (two-colourable) the limit is taken through even values of n only.

Proof. From lemma 4.1 in Soteros *et al* (1991), there exists an embedding of a graph τ in Z^3 if and only if $\tau \in \mathscr{G}_6$. Furthermore, since SAPs in H^d must have an even number of edges a graph $\tau \in \mathscr{G}_6$ can have a uniform embedding in H^3 with n edges in each branch, n odd, if and only if τ has no cycles of odd length. Thus τ has a uniform embedding in H^3 with n odd if and only if τ is bipartite (two-colourable) by König's theorem (see e.g. Roberts 1984). We first consider n even and prove equation (3.29) assuming the limit is taken only through even values of n. This will prove the theorem for the case that τ is not bipartite. Then we consider a construction which is only valid for τ bipartite and which allows us to prove the theorem for bipartite τ.

Regardless of whether n is even or odd, an upper bound for $A_N^{++}(\tau, \beta)$ can be obtained using independently embedded ω graphs. Thus we focus on obtaining a lower bound.

In order to obtain a lower bound for $A_N^{++}(\tau, \beta)$, we construct uniform embeddings of τ in H^3 with n edges in each branch. If n is even the construction consists of first finding an embedding of τ satisfying certain properties. Then we concatenate polygons in wedges to the embedding of τ to create new embeddings of τ.

A modification of the proof of lemma 4.1 in Soteros *et al* (1991), allows us to show that given any $\tau \in \mathcal{G}_6$ there exists an embedding of τ in Z^3 with the following properties:

(i) exactly one edge of each branch of τ lies in the rightmost (maximum x_1-coordinate) plane, say $x_1 = k$, of the embedding. These rightmost edges lie in the line $x_3 = 0$, $x_1 = k$, $x_2 \geqslant 0$;

(ii) each branch has an even number of edges (just divide each existing edge on the lattice into two edges); and

(iii) the edges in the line $x_3 = 0$, $x_1 = k$ are at least f edges apart, where f is the number of branches of τ.

Given any $\tau \in \mathcal{G}_6$ find an embedding η of τ in Z^3 satisfying these three properties. The edges of η in the rightmost plane are of the form $\{v, v+\hat{u}_2\}$. Represent such an edge by the vertex v. Now label the vertices v, representing edges, and the vertices of degree 1 in the rightmost plane of η with the numbers $1, \ldots, f$ in an order that increases with their x_2 coordinates. This labelling also gives a labelling of the branches of η.

Suppose η has M_i edges (note that M_i will be an even number because of the subdividing of the lattice) in the ith branch and m_i^* vertices (not including branch points) in L^3. Let m_0 be the number of branch points in L^3.

Divide the quarter-space to the right of η into f disjoint wedges. Place a $(0,1,0)$-wedge at v_1, a $(1,2,2)$-wedge at $v_2, \ldots,$ a $(i, i+1, i+1)$-wedge at $v_{i+1}, \ldots,$ and a $(f-1, f, f)$-wedge at v_f.

Concatenate to v_i an $(n - M_i + 2)$-step SAP in the ith wedge which visits L^3 $m_i + 1$ times and contains the edge $\{v_i, v_i + \hat{u}_2\}$. Delete the edge $\{v_i, v_i + \hat{u}_2\}$. If v_i is a vertex of degree 1, also delete $v_i + \hat{u}_2$ and the edge containing it. This creates an embedding of τ with n steps in each branch and with $m + f = m_0 + \Sigma_{i=1}^{f} (m_i^* + m_i)$ vertices in L^3. Thus

$$\prod_{i=1}^{f} p_{n-M_i+2, m_i}^{i-1,i}(3) \leqslant g^{++}(n, m, \tau). \tag{3.30}$$

Multiply both sides by $e^{\beta \Sigma_{i=1}^{f} m_i}$, sum over $1 \leqslant m_i \leqslant (n - M_i + 2)$ for $i = 1, \ldots, f$, take logarithms of both sides, divide by $N = nf - c + 1$ and let n go to infinity. Lemma 2 and equation (3.30) then imply

$$A^{(3)}(\beta) \leqslant \liminf_{n\to\infty} N^{-1} \log A_N^{++}(\tau, \beta). \tag{3.31}$$

For an upper bound, we fix a labelling of the branches and vertices of τ and then note that each labelled uniform embedding of τ with n edges per branch and $m+f$ vertices in L^3 (at least one in each branch) can be separated into f n-step ω graphs (each with at least one vertex in H^3) so that the number of vertices in L^3 adds to m. Since the number of ways to label the branches and branch points of an embedding of τ is bounded above by $f!2^f$, this gives

$$g^{++}(n, m, \tau) \leqslant f!2^f \sum_{m_i}{}' \prod_{j=1}^{f} g^+(n, m_j, \omega). \tag{3.32}$$

The prime on the summation indicates that the sum is over $0 \leqslant m_i \leqslant m$ for $i = 1, \ldots, f$ such that $\Sigma_{i=1}^{k} m_i = m$. Hence equation (3.28) implies

$$\limsup_{n\to\infty} N^{-1} \log A_N^{++}(\tau, \beta) \leqslant A^{(3)}(\beta). \tag{3.33}$$

Thus the theorem is proved in the case that the limit is taken only through even values of n, i.e. for τ which are not bipartite.

Suppose now that τ is bipartite. This means that we can label the vertices of τ with two labels (high and low say) so that no two vertices having the same label are joined by an edge in τ. Let n_L be the number of low vertices and hence $f-c+1-n_L$ is the number of high vertices. Arbitrarily place the low vertices along the line $x_3=1$, $x_1=1$ so that their x_2 coordinates differ by at least 7. Given an integer r, place the high vertices arbitrarily along the line $x_3=r$, $x_1=1$ so that their x_2 coordinates differ by at least 7. Since all branches of τ are between a low vertex and a high vertex, clearly r can be chosen so that an embedding of τ in H^3 can be obtained by adding edges to join appropriate low and high vertices. Furthermore, this construction can be done so that there exists a plane $x_3=j$ which intersects each branch of the embedding exactly once and so that the embedding is to the left of the plane $x_1=1$ and so that the number of edges in each branch is odd. Call the resulting embedding of τ T'. There are hence f vertices of T' in the line $x_3=j$, $x_1=1$, label these $v_0, \ldots, v_{f-1}$ according to the value of their x_2 coordinates such that v_0 has the smallest x_2 coordinate. This labelling also provides a labelling for the branches of T'. We can now concatenate a SAP to each edge $v_i+\hat{u}_3$ to create a new embedding η of τ with properties (i) and (iii) as above but now with an odd number of edges in each branch. The proof now proceeds exactly as in the non-bipartite case except that now n and the M_i, $i=1,\ldots,f$ are odd numbers.

Thus the theorem is proved for all τ. □

Corollary 1. For any $\tau \in \mathscr{G}_6$,

$$\lim_{n\to\infty} N^{-1}\log A_N^+(\tau,\beta) = A^{(3)}(\beta) \tag{3.34}$$

where if τ is not bipartite the limit is taken through even values of n only.

Proof. To prove this corollary, first note that $g^{++}(n,m,\tau) \le g^+(n,m+f-1,\tau)$ and hence equation (3.29) implies

$$A^{(3)}(\beta) \le \liminf_{n\to\infty} N^{-1}\log A_N^+(\tau,\beta) \tag{3.35}$$

with the required restriction on n if τ is not bipartite.

To obtain an upper bound we again separate a labelled embedding of τ in H^3 into f n-step ω graphs. This gives the following upper bound:

$$g^+(n,m,\tau) \le g^{++}(n,m-f+1,\tau) + \sum_{k=\max(1,m/n)}^{\min(f-1,m+1)} f!2^f[g(n,\omega)]^{f-k}\binom{f}{k}\sum_{m_i}{}' \prod_{j=1}^{k} g^+(n,\omega,m_j). \tag{3.36}$$

The prime on the second summation indicates that the sum is over $0\le m_i\le m$ for $i=1,\ldots,k$ such that $\Sigma_{i=1}^k m_i = m-k+1$. Equations (2.4), (2.8), (3.28), (3.29) and (3.36) imply

$$\limsup_{n\to\infty} N^{-1}\log A_N^+(\tau,\beta) \le A^{(3)}(\beta). \tag{3.37}$$

Equations (3.35) and (3.37) imply equation (3.34). □

Let $g^+(n,\tau)$ be the number of uniform embeddings of τ in Z^3 with n edges in each branch. $A_N^+(\tau,0) \le g^+(n,\tau)$. An upper bound for $g^+(n,\tau)$ is obtained by separating embeddings of τ into independent ω graphs. Hence corollary 1 and equation (2.1) give the following corollary.

Corollary 2. For any $\tau \in \mathscr{G}_6$

$$\lim_{n\to\infty} N^{-1} \log g^{+}(n, \tau) = \kappa_3. \tag{3.38}$$

Consider T as defined in lemma 2 and let $g^{+,\alpha,\beta}(n, m, \tau)$ be the number of uniform embeddings of τ in an (α, β, T)-wedge in H^3 with $v_b = (0, 0, 0)$, n edges per branch and $m+1$ vertices in L^3. Gaunt and Colby (1990) have shown that uniform stars in a wedge have the same growth constant as SAWs. It is now possible to show that this generalizes to uniform embeddings of any graph τ and in fact it is possible to prove the following corollary (for details of the proof see Soteros 1991).

Corollary 3. If either α or $1/\alpha$ is an integer

$$\lim_{n\to\infty} n^{-1} \log \sum_{m=0}^{n} g^{+,\alpha,\beta}(n, m, \tau)\, \mathrm{e}^{\varepsilon m} = A^{(3)}(\varepsilon). \tag{3.39}$$

Let $a^{+}(n, m, c, n_3, \ldots, n_6)$ be the number of uniform lattice animals in H^3 with n edges in each branch, $m+1$ vertices in L^3, $m \geqslant 0$, cyclomatic index c, and n_i vertices of degree $i \geqslant 3$. Let $a^{++}(n, m, c, n_3, \ldots, n_6)$ be the number of uniform lattice animals in H^3 with n edges in each branch, $m+f$ vertices in L^3 (at least one in each of the f branches), cyclomatic index c, and n_i vertices of degree $i \geqslant 3$. $N = nf - c + 1$ will be the total number of vertices in either case. An upper bound for $a^{++}(n, m, c, n_3, \ldots, n_6)$ ($a^{+}(n, m, c, n_3, \ldots, n_6)$) is easily obtained as in the proof of theorem 1 (corollary 1) by embedding f n-step ω graphs. In addition, for any $\tau \in \mathscr{G}_6$ such that τ has cyclomatic index c and vertex degrees $(n_3, \ldots, n_6)$, $g^{++}(n, m, \tau) \leqslant a^{++}(n, m\, c, n_3, \ldots, n_6) \leqslant a^{+}(n, m+f-1, c, n_3, \ldots, n_6)$. Hence we obtain the following corollary.

Corollary 4. (Zhao and Lookman 1991a) For any c and $(n_3, \ldots, n_6)$

$$\lim_{n\to\infty} N^{-1} \log \sum_{m=0}^{N} a^{++}(n, m, c, n_3, \ldots, n_6)\, \mathrm{e}^{\beta m}$$

$$= \lim_{n\to\infty} N^{-1} \log \sum_{m=0}^{N} a^{+}(n, m, c, n_3, \ldots, n_6)\, \mathrm{e}^{\beta m} = A^{(3)}(\beta). \tag{3.40}$$

For any $\tau \in \mathscr{G}_6$, consider the number of embeddings of τ (not necessarily uniform) in H^3 of size n, with $m+1$ vertices in L^3, $m \geqslant 0$, and denote this by $g^{(3)}(n, m, \tau)$.

Corollary 5. For any $\tau \in \mathscr{G}_6$

$$\lim_{n\to\infty} n^{-1} \log \sum_{m=0}^{n} g^{(3)}(n, m, \tau)\, \mathrm{e}^{\beta m} = A^{(3)}(\beta). \tag{3.41}$$

Proof. The proof of this follows the proof of theorem 1 except now one need only concatenate one polygon into the quarter-space to the right of the initial embedding. This gives a lower bound and as usual the upper bound is obtained by separating embeddings into independent ω graphs. $\square$

Let $a(n, m, c, n_3, \ldots, n_6)$ be the number of lattice animals in H^3 with n vertices, $m+1$ vertices in L^3, $m \geq 0$, cyclomatic index c, and n_i vertices of degree $i \geq 3$. Let f be the number of branches. An upper bound for these is easily obtained as above by embedding f n-step ω graphs. In addition, for any $\tau \in \mathscr{G}_6$ such that τ has cyclomatic index c and vertex degrees $(n_3, \ldots, n_6)$, $g^{(3)}(n, m, \tau) \leq a(n, m, c, n_3, \ldots, n_6)$. Hence we obtain the following corollary.

Corollary 6. For any c and $(n_3, \ldots, n_6)$

$$\lim_{n\to\infty} n^{-1} \log \sum_{m=0}^{n} a(n, m, c, n_3, \ldots, n_6)\, \mathrm{e}^{\beta m} = A^{(3)}(\beta). \tag{3.42}$$

4. The square lattice results

In the previous section, theorem 1 and corollaries 1, 5 and 6 indicate that the free energy for embeddings interacting with a surface in H^3 is independent of the specified topology and independent of whether the embeddings are uniform (theorem 1 and corollary 1) or unrestricted (corollaries 5 and 6). However, for $d = 2$ the results are not independent of topology and the results for uniform embeddings are quite different from those for unrestricted embeddings. Since the results for unrestricted embeddings are easier to obtain we start this section by looking at this case and then consider uniform embeddings.

First we need to determine which graphs have embeddings in Z^2. Since non-planar graphs are not embeddable in Z^2 not all graphs in $\mathscr{G}_4$ have embeddings in Z^2. However, if $\mathscr{G}_\mathrm{p}$ is the subset of $\mathscr{G}_4$ consisting only of planar graphs then for any graph $\tau \in \mathscr{G}_\mathrm{p}$ there exists an embedding of τ in Z^2. In particular an embedding of $\tau \in \mathscr{G}_\mathrm{p}$ can be constructed so that given a labelling of the branches of τ with the integers $1, \ldots, f$ and an integer j, $1 \leq j \leq f$, there exists an embedding of τ in Z^2 such that:

(i) the embedding is confined to H^2;

(ii) the jth branch of τ contains the top vertex of the embedding and the top vertex, v_t, is either in L^2 (if the jth branch ends in a vertex of degree one) or $v_\mathrm{t} - \hat{u}_2$ is in L^2;

(iii) there exists a line $x_1 = k$ which cuts every branch of the embedding exactly twice and the vertices in this line are at least $2f$ edges apart, where f is the number of branches of τ.

If τ is a tree the result can be proved following an argument similar to that of the proof of lemma 4.1 in Soteros *et al* (1991). For $\tau \in \mathscr{G}_\mathrm{p}$ with cyclomatic index $c > 0$, we find the appropriate embedding as follows. There exist c edges of τ, $e_1, \ldots, e_c$, such that when they are cut the resulting graph is a tree. Let n_1 be the number of vertices of degree 1 in τ. Consider the graph $\eta \in \mathscr{G}_\mathrm{p}$ which is obtained from τ by cutting the edges $e_1, \ldots, e_c$ so that η is a tree and has $n_1 + 2c$ vertices of degree 1. Consider any planar embedding of τ in R^2. Cutting the branches corresponding to $e_1, \ldots, e_c$ in the embedding gives a planar embedding of η in R^2. We use this planar embedding of η to define a labelling of the branches of η. The branches are labelled so that when the construction of lemma 4.1 of Soteros *et al* (1991) is applied to obtain an embedding of η in Z^2 the resulting embedding can be extended to an embedding of τ in Z^2 satisfying properties (i)–(iii) above. For more details of this construction see Soteros (1991).

Now that we know how to construct an embedding for any $\tau \in \mathscr{G}_\mathrm{p}$ we get the following result.

Theorem 2. For any $\tau \in \mathscr{G}_p$

$$\lim_{n\to\infty} n^{-1} \log g^{(2)}(n, \tau) = \kappa_2. \tag{4.1}$$

The proof of this follows the proof of equation (2.5) except that the lower bound is now obtained by starting with an embedding of τ in Z^2. □

Now, for any $\tau \in \mathscr{G}_p$, consider the number of embeddings of τ (not necessarily uniform) in H^2 of size n with $m+1$ vertices in L^2, $m \geqslant 0$, and denote this by $g^{(2)}(n, m, \tau)$. We ask how the free energy function obtained from $g^{(2)}(n, m, \tau)$ depends on τ. For $d=3$ the answer to the analogous question is that the free energy is equal to that for SAWs (corollary 5) independent of the choice of τ; for $d=2$ we find the free energy depends on the choice of τ.

As in the previous section, we start by looking at the case $\tau = \pi$ and hence the embeddings are SAPs. We can prove that the free energy for SAPs is not always equal to the free energy for SAWs. The proof of this is similar to that given in Whittington and Soteros (1991) for uniform 3-stars in Z^2; however, in this case we can prove the existence of the free energy.

To show that the free energy exists, let $p^*_{n,m}$ be the number of distinct polygons in H^2 with $m+1$ vertices in L^2 two of which are v_b and $v_t - \hat{u}_2$. For these polygons

$$p^*_{n,k}\, p^*_{m,j} \leqslant p^*_{n+m,k+j} \tag{4.2}$$

and this implies that

$$B^*_n(\beta) B^*_m(\beta) \leqslant (n+m+1) B^*_{n+m}(\beta) \tag{4.3}$$

where $B^*_n(\beta) = \Sigma^n_{m=0}\, p^*_{n,m}\, e^{\beta m}$. Thus by the theory of subadditive functions the free energy function for these polygons will exist, call it $B(\beta)$, and hence

$$B(\beta) = \lim_{n\to\infty} \sum_{m=0}^{n} n^{-1} \log B^*_n(\beta). \tag{4.4}$$

We now show that all SAPs with at least one vertex in L^2 have free energy $B(\beta)$. The result is the following.

Lemma 4.

$$\lim_{n\to\infty} n^{-1} \log B^{++(2)}_n(\beta) = \lim_{n\to\infty} n^{-1} \log B^{+(2)}_n(\beta) = B(\beta) \tag{4.5}$$

with $B^{++(2)}_n$ and $B^{+(2)}_n$ as in equations (3.2) and (3.3) respectively.

Proof. Since $p^*_{n,m} \leqslant p^{++}_{n,m}(2)$ and since polygons with v_b in L^2 can be concatenated in pairs (according to the x_2 coordinate of v_t) to form polygons with v_b and $v_t - \hat{u}_2$ in L^2 one can show that

$$B(\beta) = \lim_{n\to\infty} n^{-1} \log B^{++(2)}_n(\beta) = \lim_{n\to\infty} n^{-1} \log \sum_{m=0}^{n/2-1} p^{++}_{n,m}(2)\, e^{\beta m}. \tag{4.6}$$

Hammersley and Whittington's (1985) argument concerning SAPs in wedges yields an upper bound for all SAPs in H^2 in terms of SAPs in a $(0, \alpha, 0)$-wedge with v_b in L^2. Thus by extending their argument one can obtain an upper bound for $p^+_{n,m}(2)$ in terms of $p^{++}_{n,m}(2)$. Further, since $p^{++}_{n,m}(2) \leqslant p^+_{n,m}(2)$, it can be proved that

$$B(\beta) = \lim_{n\to\infty} n^{-1} \log B^{+(2)}_n(\beta) = \lim_{n\to\infty} n^{-1} \log \sum_{m=0}^{n/2-1} p^{+}_{n,m}(2)\, e^{\beta m}. \tag{4.7}$$

□

Now that we know the free energy, $B(\beta)$, exists for SAPs we can ask how it is related to the free energy for SAWs, $A^{(2)}(\beta)$. We show in fact that there exists $\beta_0 > 0$ such that for all $\beta \geq \beta_0$, $B(\beta) < A^{(2)}(\beta)$ while for $\beta < \beta_0$, $B(\beta) = A^{(2)}(\beta) = \kappa_2$.

For $d = 2$ the maximum number of vertices a polygon can have in L^2 is $n/2$. Let $p_n^{++}(2) = \Sigma_{m=0}^{n/2-1} p_{n,m}^{++}(2)$. For $\beta \leq 0$,

$$p_{n-2}^{++}(2)\,\mathrm{e}^{2\beta} \leq p_{n,2}^{++}(2)\,\mathrm{e}^{2\beta} \leq B_n^{++(2)}(\beta) \leq B_n^{++(2)}(0) \leq p_n^{++}(2). \tag{4.8}$$

The first inequality on the left comes from the fact that any polygon in H^2 with v_b in L^2 can be translated up by $\hat{u}_2$ and then converted into a polygon with exactly two vertices in L^2 (add the following edges and corresponding vertices, $\{v_b, v_b - \hat{u}_2\}$, $\{v_b - \hat{u}_2, v_b - \hat{u}_2 + \hat{u}_1\}$, $\{v_b - \hat{u}_2 + \hat{u}_1, v_b + \hat{u}_1\}$ and delete $\{v_b, v_b + \hat{u}_1\}$). Thus for all $\beta \leq 0$, $B(\beta) = \kappa_2$.

For $\beta > 0$, $B_n^{(2)}(0) \leq B_n^{(2)}(\beta)$ and

$$\mathrm{e}^{(n/2-1)\beta} \leq B_n^{++(2)}(\beta) \leq p_n^{++}(2)\,\mathrm{e}^{(n/2-1)\beta}. \tag{4.9}$$

Thus for all β,

$$\max(\kappa_2, \beta/2) \leq B(\beta) \leq \max(\kappa_2, \kappa_2 + \beta/2) \tag{4.10}$$

and for sufficiently large β the upper bound here is smaller than the lower bound in equation (2.8) for $A^{(2)}(\beta)$. Thus for sufficiently large β, $B(\beta) < A^{(2)}(\beta)$. Furthermore, equation (4.10) implies that $B(\beta)$ is non-analytic and hence there is a phase transition in the model for some β_0, $0 \leq \beta_0 \leq 2\kappa_2$. Since the free energy for SAPs is bounded above by the free energy for SAWs $\beta_0 \geq \beta_c$ and hence the adsorption temperature (proportional to $1/\beta$) for SAPs is at least as low as that for SAWs. □

Now that we have characterized the free energy for SAPs and the free energy for SAWs and the relationship between them we can ask if there is a relationship between the free energy for embeddings of a graph τ and either $B(\beta)$ or $A^{(2)}(\beta)$. The answer for the case that the embeddings are unrestricted is as follows.

Theorem 3. Consider any $\tau \in \mathcal{G}_p$. Let $A_n(\tau, \beta) = \Sigma_{m=0}^{n} g^{(2)}(n, m, \tau)\,\mathrm{e}^{\beta m}$. If τ has a cut edge,

$$\lim_{n\to\infty} n^{-1} \log A_n(\tau, \beta) = A^{(2)}(\beta). \tag{4.11}$$

Depending on τ the limit may be taken through only even values of n.

If τ does not have a cut edge then

$$\lim_{n\to\infty} n^{-1} \log A_n(\tau, \beta) = B(\beta). \tag{4.12}$$

Proof. Suppose τ has a cut edge. Therefore the edge can be cut and τ can be separated into two graphs, $\tau_1 \in \mathcal{G}_p$ and $\tau_2 \in \mathcal{G}_p$, each of which has a vertex of degree 1, v_1 and v_2 respectively, such that when they are joined τ is formed. Find an embedding η_1 of τ_1 in Z^2 satisfying properties (i)–(iii) and so that $v_t = v_1$. Find an embedding η_2 of τ_2 in Z^2 satisfying properties (i)–(iii) and so that $v_t = v_2$. Suppose η_1 has M_1 vertices m_1 of which are in L^2 and η_2 has M_2 vertices m_2 of which are in L^2. Concatenate to v_1 the first step of an $(n - M_1 - M_2 + 1)$-step (*)-walk starting and ending in L^2 and with $m+1$ vertices in L^2. Translate and reflect η_2 so that v_2 can be concatenated to the last step of the walk. The result is an embedding of τ of size n with $M = m_1 + m_2 + m - 1$ vertices in L^2. Denote the number of n-step (*)-walks with $m+1$ vertices in L^2 by $c_{n,m}^*(2)$. Hence

$$c_{n-M_1-M_2+1,m}^*(2) \leq g^{(2)}(n, M, \tau). \tag{4.13}$$

An upper bound can be obtained by independently embedding walks for the branches of τ.

Suppose τ does not have a cut edge. In the special case that all branch points of τ are degree 4 then τ is Eulerian (i.e. there is a chain which goes through every edge of τ exactly once and returns to where it started). Let n_4 be the number of branch points of τ. Each branch point will be traversed twice by the Eulerian closed chain and τ can thus be separated into a set of cycles none of which share an edge of τ. Let k be the number of cycles. Hence an upper bound for $g^{(2)}(n, m, \tau)$ can be obtained by embedding k independent polygons such that the jth polygon has $4 \leqslant n_j \leqslant (n+n_4-4k)$ vertices, $0 \leqslant m_j \leqslant n_j/2$ of which are in L^2, with $\Sigma_{j=1}^k n_j = n+n_4$ and $\Sigma_{j=1}^k m_j = m+1$. Therefore

$$\limsup_{n\to\infty} n^{-1} \log A_n(\tau, \beta) \leqslant B(\beta). \tag{4.14}$$

A lower bound for $g^{(2)}(n, m, \tau)$ can be obtained in terms of polygons as in the proof of corollary 5. This gives

$$B(\beta) \leqslant \liminf_{n\to\infty} n^{-1} \log A_n(\tau, \beta). \tag{4.15}$$

Equations (4.15) and (4.14) imply equation (4.12).

Suppose that τ does not have a cut edge and is not Eulerian. In this case equation (4.12) still holds. To prove this, note that any embedding of τ in H^2 will have a boundary which forms a connected subgraph of τ and is Eulerian. The branches not in the boundary cannot have more than two vertices (their branch ends) in L^2. Let $\mathscr{T}$ be the set of Eulerian subgraphs of τ and for any β define $b(\beta) = \max\{e^\beta, 1\}$. Hence

$$A_n(\tau, \beta) \leqslant \sum_{\eta\in\mathscr{T}} \sum_{m_i}{}' A_{m_1}(\eta, \beta)(f_\tau - f_\eta)!2^{f_\tau - f_\eta} \prod_{i=2}^{f_\tau - f_\eta + 1} g^{(2)}(m_i, \omega) b(\beta) \tag{4.16}$$

where the prime on the summation indicates that the sum is over all $1 \leqslant m_i \leqslant n$ for $i = 1, \ldots, f_\tau - f_\eta + 1$ such that $\Sigma_{i=1}^{f_\tau - f_\eta + 1} m_i = n + f_\tau - f_\eta + c_\tau - c_\eta$, f_τ and c_τ are respectively the number of branches and cycles in τ and f_η and c_η are the number of branches and cycles in η. There are

$$\binom{n+f_\tau - f_\eta + c_\tau - c_\eta - 1}{f_\tau - f_\eta}$$

terms in the primed sum. Let η^* be the value of η for which the primed sum is maximal and let $\{m_i^*, i = 1, \ldots, f_\tau - f_{\eta^*} + 1\}$ be the set of m_i for which the general term in the primed sum is maximal. We then get an upper bound on the right term in equation (4.16) and thus

$$\begin{aligned} A_n(\tau, \beta) \leqslant |\mathscr{T}| &\binom{n+f_\tau - f_{\eta^*} + c_\tau - c_{\eta^*} - 1}{f_\tau - f_{\eta^*}} \\ &\times A_{m_1^*}(\eta^*, \beta)(f_\tau - f_{\eta^*})!2^{f_\tau - f_{\eta^*}} \prod_{i=2}^{f_\tau - f_{\eta^*}+1} g^{(2)}(m_i^*, \omega) b(\beta). \end{aligned} \tag{4.17}$$

Taking logarithms, dividing by n, letting $n \to \infty$ and using equations (4.14), (4.1) and (4.10) gives

$$\lim_{n\to\infty} n^{-1} \log A_n(\tau, \beta) \leqslant B(\beta). \tag{4.18}$$

This along with equation (4.15) implies equation (4.12). Thus the theorem is proved. $\square$

If now the topology of the polymer is only specified by the cyclomatic index and degree indices instead of by a specific graph τ then we get the following result. Let $a(n, m, c, n_3, n_4)$ be the number of lattice animals in H^2 with n vertices, $m+1$ vertices in L^2, $m \geqslant 0$, cyclomatic index c, and n_i vertices of degree $i \geqslant 3$. This case is analogous to the case discussed in corollary 6 for $d=3$; however, in two dimensions the results depend on τ.

Corollary 7. Consider any c and (n_3, n_4) and animals restricted to H^2. If either $n_3 > 0$ or $c \neq n_4+1$ and if there exists a $\tau \in \mathscr{G}_p$ with cyclomatic index c and degree set (n_3, n_4), then

$$\lim_{n\to\infty} n^{-1} \log \sum_{m=0}^{n} a(n, m, c, n_3, n_4)\, e^{\beta m} = A^{(2)}(\beta). \tag{4.19}$$

If $c = n_4+1$ and $n_3 = 0$ then

$$\lim_{n\to\infty} n^{-1} \log \sum_{m=0}^{n} a(n, m, c, 0, n_4)\, e^{\beta m} = B(\beta). \tag{4.20}$$

Proof. The proof of this relies on the fact that if $c \neq n_4+1$ then either $n_1 > 0$ or $n_3 > 0$. In either case there is a $\tau \in \mathscr{G}_p$ with cyclomatic index c and vertex degree set (n_3, n_4) which has a cut edge. (In particular, if $n_1 = 0$, $n_4 = 0$, then we must have $n_3 = 2(c-1)$ and $f = 3(c-1)$ and a graph which satisfies this is formed by an alternating chain of circles and lines starting and ending with a circle so that the simplest case $(c=2)$ is the dumbbell graph. If $n_1 = 0$, $n_4 > 0$ then $n_3 = 2(c-1) - 2n_4$ and $f = 3(c-1) - n_4$ and such a graph can be obtained from the alternating chain just described by removing n_4 of the lines.) One can thus obtain a lower bound for $a(n, m, c, n_3, n_4)$ as in the proof of theorem 3. An upper bound is obtained using independently embedded ω graphs. If $c = n_4+1$ and $n_3 = 0$ then $n_1 = 0$ and the proof is exactly the same as the proof leading to equation (4.12). □

The last two results show how the free energy of embeddings with specified topology in Z^2 are related to the free energies for SAWs and SAPs in the case that the embeddings are unrestricted. We now try and determine if there is a corresponding relationship in the special case that the embeddings are uniform. We must, however, first determine the growth constant for uniform embeddings of a graph τ in Z^2.

For any $\tau \in \mathscr{G}_p$, let $g^+(n, \tau)$ be the number of uniform embeddings of τ in Z^2 with n edges in each branch. While the free energy for the uniform case will be shown to be dependent on τ we find that the growth constant of $g^+(n, \tau)$ is independent of τ. This gives a new result for $d=2$ which is analogous to corollary 2 for $d=3$; however, the proof of this result is considerably different from the proof of corollary 2.

Theorem 4. If $\tau \in \mathscr{G}_p$, $N = nf - c + 1$, then

$$\lim_{n\to\infty} N^{-1} \log g^+(n, \tau) = \kappa_2 \tag{4.21}$$

where if τ is not bipartite the limit is taken only through even values of n.

Proof. For $\tau = \omega$ or $\tau = \pi$ the result follows from equations (2.1) and (2.2) respectively.

As in the proof of theorem 1, we again consider the case that τ is not bipartite first. For $\tau \neq \omega$, π and τ not bipartite, the proof basically consists of using an embedding of τ with properties (i)–(iii). This embedding can be made so that it is uniform and then a lower bound for $g^+(n, \tau)$ is obtained by concatenating walks in wedges to the vertices of τ in the line $x_1 = k$ to create a new embedding of τ. An upper bound is obtained using independently embedded ω graphs.

Consider an embedding of τ in Z^2 satisfying the properties (i)–(iii). The line $x_1 = k$ cuts every branch of τ exactly twice. Label the branches of this embedding arbitrarily with the integers $1, \ldots, f$ and let m_i be the number of edges in the ith branch. Note that we can easily assume that the m_i are even and divisible by 4 (if not just subdivide each edge of the lattice twice). Let $m^* = \max_i\{m_i\}$ and $m_* = \min_i\{m_i\}$. If $m^* = m_*$, do nothing. Otherwise, at each vertex of the ith branch in the line $x_1 = k$ insert a walk which has $m^* - (m_* + m_i)/2$ steps as follows $\{\hat{u}_1, (m^* - m_i/4)[\hat{u}_2, \hat{u}_1, -\hat{u}_2], (m^* - 2m_* + m_i - 4/4)\hat{u}_1\}$. This results in a uniform embedding, T, of τ in Z^2 with each branch having $2m^* - m_*$ edges and such that the line $x_1 = k$ cuts each branch exactly twice.

There are hence $2f$ vertices of T in the line $x_1 = k$, label these $v_0, \ldots, v_{2f-1}$ according to the value of their x_2 coordinates such that v_0 has the smallest x_2 coordinate.

Given any integers $x \geq 0$ and $j \geq 0$, define a $(j; x)$-ladder to be the unique $(j+1)x$-SAW which starts at the origin, goes j steps in the x_2 direction, one step in the x_1 direction, and then repeats this pattern a total of x times. Define a $(j; x)$-tooth to be the unique $2(j+1)x$-SAW which starts at the origin, goes one step in the x_1 direction, goes j steps in the x_2 direction, one step in the x_1 direction, then j steps in the $-x_2$ direction, and repeats this pattern at total of x times. Hence a $(j; x)$-ladder is contained in a $(j, j+1, j+1)$-wedge and ends at the vertex (x, jx) and a $(j; x)$-tooth is contained in a slit of height j and ends at the vertex $(2x, 0)$.

Let $c_n^{1,M,2} \equiv A_n^{1,M,2}(0) = \sum_{m=1}^{n} c_{n,m}^{1,M,2}$ (see equation (3.9)) be the number of rooted n-SAWs in W_M^1. Let $c_n^{1,2} = \sum_{M=0}^{n} c_n^{1,M,2}$. Equation (3.11) gives

$$\lim_{n\to\infty} n^{-1} \log c_n^{1,2} = \kappa_2. \tag{4.22}$$

Given an integer $\hat{n}$ there exists an M such that

$$c_{\hat{n}}^{1,M,2} \geq \frac{c_{\hat{n}}^{1,2}}{\hat{n}}. \tag{4.23}$$

Since there are $(\lfloor M/2 \rfloor + 1)(\lceil M/2 \rceil + 1)$ sites in W_M^1, $M \geq \sqrt{\hat{n}}$. Given an integer $\hat{n}$ such that $\sqrt{\hat{n}} > 2f - 1$ and hence $M \geq 2f - 1$, fix an M as in equation (4.23). Let $C_M = 2Mf(2f+1)$. Now, given any even integer $n \geq 2C_M + 2m^* - m_*$ there exist positive integers p and q such that $n = 2C_M + 2p\hat{n} + 2m^* - m_* + 2q$. We can now construct a uniform embedding of τ with n edges in each branch as follows. (This construction was suggested by Madras (1991).) To do this first split T into two parts by dividing it along the line $x_1 = k$ and letting v_i^{L} be the vertex v_i in T^{L}, the part of T to the left of $x_1 = k$, and v_i^{R} be the vertex v_i in T^{R}, the part of T to the right of $x_1 = k$ (any edges in the line $x_1 = k$ appear only in T^{R}). Translate T^{R} so that $v_i^{\mathrm{R}} = v_i^{\mathrm{L}} + [(4f+p)M + q]\hat{u}_1$ for each i.

Concatenate a $(j; M)$-ladder to v_j^{L} and concatenate a $(j; M)$-ladder reflected through the x_2 axis to v_j^{R}, for $2f - 1 \geq j \geq 0$. The end points of the two $(j; M)$-ladders can now be connected by concatenating a sequence of $(l; M)$-tooth walks, $l = 2f - 1, 2f - 2, \ldots, 0$, $l \neq j$ then concatenating p $\hat{n}$-SAWs each in W_M^1 and finally concatenating q steps in the x_1 direction. The result is a uniform embedding of τ with the specified number of edges in each branch. Figure 2 illustrates the case $f = 2$, $M = 4$, $q = 0$ and $p > 6$. The above construction implies that

$$[c_{\hat{n}}^{1,M,2}]^{2fp} \leq g^+(2C_M + 2p\hat{n} + 2m^* - m_* + 2q, \tau) = g^+(n, \tau). \tag{4.24}$$

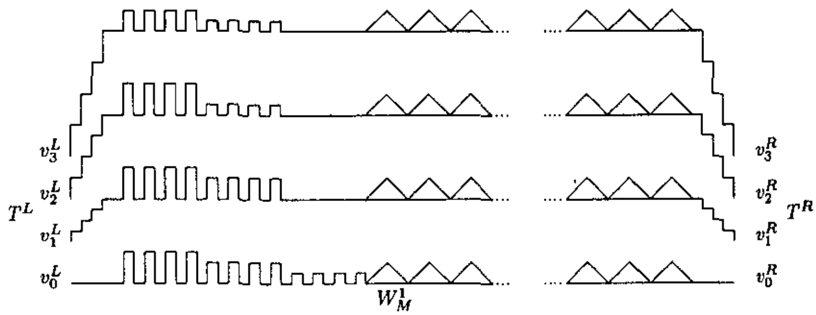

Figure 2. This figure illustrates how the wedges W_M^1 are concatenated to the v_i for the constructions in theorem 4. The case that $M=4$, $q=0$ and $p>6$ is shown.

Taking logarithms, dividing by $N \equiv fn-c+1$, fixing $\hat{n}$ and letting $n \to \infty$ in equation (4.24) yields

$$\hat{n}^{-1} \log c_{\hat{n}}^{1,M,2} \leqslant \liminf_{n\to\infty} N^{-1} \log g^{+}(n, \tau). \tag{4.25}$$

Using equation (4.23) and letting $\hat{n} \to \infty$ leads to

$$\liminf_{n\to\infty} N^{-1} \log g^{+}(n, \tau) \geqslant \kappa_2. \tag{4.26}$$

An upper bound in terms of independently embedded ω graphs can then be used to obtain equation (4.21).

Suppose τ is bipartite. Given any planar representation of τ, label the vertices of τ with the two labels left and right so that no edge joins two vertices with the same label. Draw a straight line so that all the right vertices are to the right of the line and now 'pull' the left vertices (allow the length of the edges to grow while maintaining the planarity of the embedding) so that they are all to the left of the line. This can be done so that the line cuts every edge of τ exactly once. By replacing the edges of this embedding by appropriate walks it is possible to obtain an embedding of τ in H^2 so that for some k the line $x_1 = k$ cuts each branch of the embedding exactly once. Now we can proceed just as above only now $\hat{n}$ can be either even or odd and there are only f vertices in $x_1 = k$. The upper bound can again be obtained using independently embedded ω graphs and thus the proof is complete. □

Now that we know what the growth constant for the uniform embeddings of a graph τ is we can next study the associated free energy. We first review the result of Whittington and Soteros for uniform stars and then use their result to get a result for general τ.

Let the tree graph in $\mathscr{G}_p$ with one vertex of degree f and f vertices of degree 1 be called the f-star graph and denote it by σ_f. In two dimensions the maximum number of vertices that a uniform embedding (n edges in each branch) of σ_f ($f=3$ or 4) can have in L^2 is $2n+1$, while the total number of vertices is $N=nf+1$. For any $\tau \in \mathscr{G}_p$, let $g^{+}(n, m, \tau)$ be the number of uniform embeddings of τ in H^2 with n edges in each branch and $m+1$ vertices in L^2, $m \geqslant 0$. Thus, from the argument of Whittington and Soteros (1991), we have the following.

Lemma 5. For $f=3$ or 4 and $N=nf+1$ let $A_N^+(\sigma_f,\beta)=\Sigma_{m=0}^{2n+1} g^+(n,m,\sigma_f)\,e^{\beta m}$.

$$\max(\kappa_2, 2\beta/f) \leqslant \liminf_{n\to\infty} N^{-1}\log A_N^+(\sigma_f,\beta)$$
$$\leqslant \limsup_{n\to\infty} N^{-1}\log A_N^+(\sigma_f,\beta) \leqslant \varsigma\nabla\max(\kappa_2, \kappa_2+2\beta/f). \tag{4.27}$$

Using lemma 5 and theorem 4 we obtain the following result.

Theorem 5. (Uniform embeddings) For any $\tau\in\mathscr{G}_p$, let $A_N^+(\tau,\beta)=\Sigma_{m=0}^{n} g^+(n,m,\tau)\,e^{\beta m}$, $N=nf-c+1$. Let $r=f-c+n_3+n_4+1/2f$ if τ has a cut edge and let $r=\min\{f-c+n_3+n_4+1/2f,\frac{1}{2}\}$ otherwise. Note that $r<1$ if $\tau\neq\omega$. Let $s=1$ if τ is bipartite and $s=2$ otherwise. Then

$$\max\left(\kappa_2, \frac{\beta}{sf}\right) \leqslant \liminf_{n\to\infty} N^{-1}\log A_N^+(\tau,\beta)$$
$$\leqslant \limsup_{n\to\infty} N^{-1}\log A_N^+(\tau,\beta) \leqslant \max(\kappa_2, \kappa_2+r\beta). \tag{4.28}$$

If τ is not bipartite the limits are taken through only even values of n. Therefore the free energy for $\tau\neq\omega$ is not equal to $A^{(2)}(\beta)$.

Proof. Consider $\tau\in\mathscr{G}_p$ with f branches, cyclomatic index c and n_j vertices of degree j, $j=1,3,4$. If $\tau=\omega$ then equation (2.8) gives the result. If $\tau=\pi$ then equation (4.10) gives the result.

Assume $\tau\neq\omega$ and $\tau\neq\pi$, i.e. either $n_3>0$ or $n_4>0$. Consider any uniform embedding η of τ in H^2 with n edges in each branch and m vertices in L^2. If n is odd then remove the middle edge of every branch of η. This results in n_1 ω graphs each with $(n-1)/2+1$ vertices, n_3 uniform 3-stars each with $3(n-1)/2+1$ vertices and n_4 uniform 4-stars each with $2n-1$ vertices. The ω graphs can have at most $(n-1)/2+1$ vertices in L^2 and the 3-stars and 4-stars can have at most n vertices in L^2. Hence $m\leqslant n_1(n+1)/2+(n_3+n_4)n=n(f-c+n_3+n_4+1)/2+n_1/2=m^*(n)$. If n is even then cut each branch at the middle vertex to create n_1 ω graphs each with $n/2+1$ vertices, n_3 uniform 3-stars each with $3n/2+1$ vertices and n_4 uniform 4-stars each with $2n+1$ vertices. The ω graphs can have at most $n/2+1$ vertices in L^2 and the 3-stars and 4-stars can have at most $n+1$ vertices in L^2. Hence $m\leqslant n_1(n+2)/2+(n_3+n_4)(n+1)=n(f-c+n_3+n_4+1)/2+n_1+n_3+n_4=m^*(n)$.

Hence for $\beta\geqslant 0$

$$A_N^+(\tau,\beta)\leqslant g^+(n,\tau)\,e^{\beta m^*(n)}. \tag{4.29}$$

For $\beta<0$

$$A_N^+(\tau,\beta)\leqslant g^+(n,\tau). \tag{4.30}$$

In the special case that τ does not have a cut edge $A_N^+(\tau,\beta)\leqslant A_N(\tau,\beta)$ and equation (4.12) imply that

$$\limsup_{N\to\infty} N^{-1}\log A_N^+(\tau,\beta)\leqslant B(\beta)\leqslant\max(\kappa_2,\kappa_2+\beta/2). \tag{4.31}$$

Hence taking logarithms, dividing by N and letting n go to infinity in equations (4.29) and (4.30) combined with equation (4.31) gives the result.

The construction used in the proof of theorem 4 gives the lower bounds. In particular, consider a uniform embedding of τ constructed as in figure 2 however with $\hat{n}=M$, we thus get

$$e^{\beta(Mp+q+M_0)} \leq g^{+}(n, pM+M_0, \tau)\, e^{\beta(Mp+q+M_0)} \leq A_N^{+}(n, \tau) \tag{4.32}$$

where $n=sC_M+spM+2m^*-m_*+sq$ ($s=1$ if τ is bipartite and $s=2$ otherwise), $N=nf-c+1$ and $M_0+Mp+q+1$ is the number of vertices of the embedding in L^2. Taking logarithms, dividing by N and letting n go to infinity (with M fixed) in equation (4.32) gives the β/sf factor in the lower bound.

On the other hand, it is possible to obtain a uniform embedding of τ constructed as in figure 2 with M satisfying equation (4.23) and so that the embedding has exactly two vertices in L^2. In this case equation (4.26) becomes

$$\kappa_2 \leq \liminf_{n\to\infty} N^{-1} \log g^{+}(n, 1, \tau). \tag{4.33}$$

Note that

$$g^{+}(n, 1, \tau)\, e^{\beta} \leq A_N^{+}(\tau, \beta). \tag{4.34}$$

Therefore taking logarithms, dividing by $N=nf-c+1$ and letting n go to infinity in equation (4.34) combined with equation (4.33) gives the κ_2 factor in the lower bound.

A result similar to theorem 5 can be obtained for uniform lattice animals in H^2 restricted to having cyclomatic index c and n_i vertices of degree i, $i=3,4$.

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